Math 1510 Week 7

Relationship between Derivatives and Graph

Monotonicity

Theorem Let I be an interval.

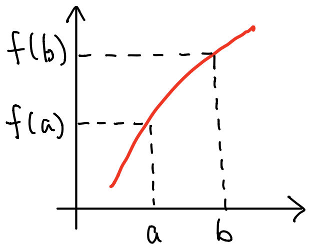
$f(x)$ is differentiable on I

If $f'(x)\begin{cases} \equiv 0 \\ \geq 0 \\ \leq 0 \end{cases}$ on I,

then $f(x)$ is $\begin{cases} \text{constant} \\ \text{increasing} \\ \text{decreasing} \end{cases}$ on I

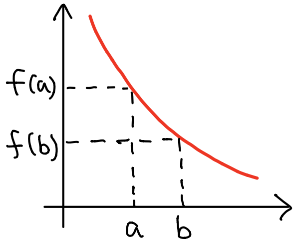
Rmk Increasing means

$a < b \Rightarrow f(a) \leq f(b)$



Decreasing means

$a < b \Rightarrow f(a) \geq f(b)$



Pf of Case 1 : $f'(x) \equiv 0$ on I

Suppose $a, b \in I$ with $a < b$

By Lagrange's MVT, $\exists c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c) = 0 \quad (\because c \in I)$$

$\Rightarrow f(b) - f(a) = 0 \Rightarrow f(a) = f(b)$

Hence f is a constant function

Pf of Case 2 $f'(x) \geq 0$ on I

Suppose $a, b \in I$ with $a < b$

By Lagrange's MVT, $\exists c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c) \geq 0 \quad (\because c \in I)$$

$\Rightarrow f(b) - f(a) \geq 0 \Rightarrow f(a) \leq f(b)$

Hence f is increasing

Pf of Case 3 is similar

eg Show that $\arctan x \leq x$ for $x \geq 0$.

Sol Let $f(x) = \arctan x - x$

Then for $x \geq 0$,

$$f'(x) = \frac{1}{1+x^2} - 1 \leq \frac{1}{1+0^2} - 1 = 0$$

$\Rightarrow$ $f(x)$ is decreasing on $[0, \infty)$

$\therefore$ $f(x) \leq f(0) = 0$ for $x \geq 0$

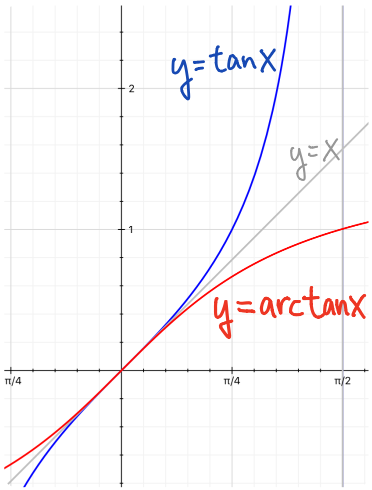
i.e. $\arctan x \leq x$

Similar Exercise

Show that

$$\tan x \geq x$$

for $0 \leq x < \frac{\pi}{2}$



eg Show that

$$f(x) = \arcsin x + \arccos x$$

is a constant function on $(-1, 1)$.

Sol For $x \in (-1, 1)$,

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$$

$\therefore$ $f(x)$ is a constant function on $(-1, 1)$

Rmk

$$f(0) = \arcsin 0 + \arccos 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

f is constant $\Rightarrow$ $f(x) \equiv \frac{\pi}{2}$ on $(-1, 1)$

Ex Show that

① $e^x \geq x \geq \ln(1+x)$ for $x \geq 0$

② $x \geq \sin x$ for $x \geq 0$

$x \leq \sin x$ for $x \leq 0$

Concavity

Defn

$f(x)$ is said to be concave up (down) if the line segment between any 2 points of its graph is above (below) the graph.

A point of inflection is where the graph changes concavity.

Theorem Let I be an interval.

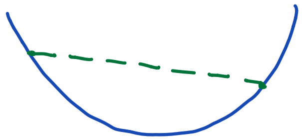
$f(x)$ is twice differentiable (i.e. $f''(x)$ exists) on I

If $\begin{cases} f''(x) \geq 0 \\ f''(x) \leq 0 \end{cases}$ on I

then $f(x)$ is $\begin{cases} \text{concave up} \\ \text{concave down} \end{cases}$ on I

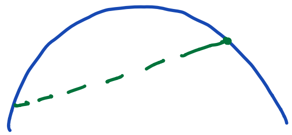
Picture:

$f'' \geq 0$ (eg. x^2)



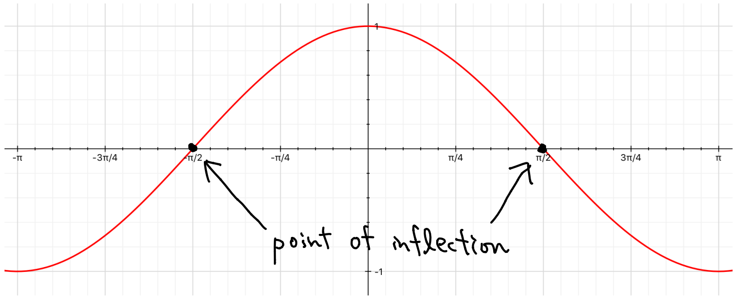
Concave up

$f'' \leq 0$ (eg. $-x^2$)



Concave down

eg $f(x) = \cos x$, $f''(x) = -\cos x = -f(x)$



$f'' > 0$	$f'' < 0$	$f'' > 0$
concave up	concave down	concave up

Relative Extrema and Derivative Tests

Recall Let $c \in D_f$. $f(x)$ is said to have

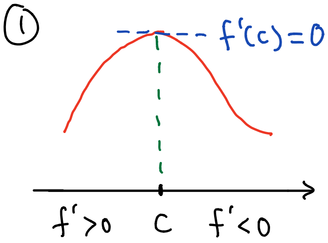
$$\text{relative} \begin{cases} \text{maximum} \\ \text{minimum} \end{cases} \text{at } c \text{ if } \begin{cases} f(x) \leq f(c) \\ f(x) \geq f(c) \end{cases} \text{near } c$$

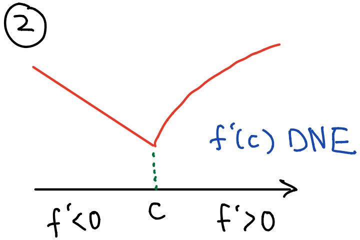
First Derivative test

Let $f(x)$ be continuous at c and $a < c < b$.

① If $f'(x) > 0$ on (a,c), $f'(x) < 0$ on (c,b) then f has a relative maximum at c.

② If $f'(x) < 0$ on (a,c), $f'(x) > 0$ on (c,b) then f has a relative minimum at c.





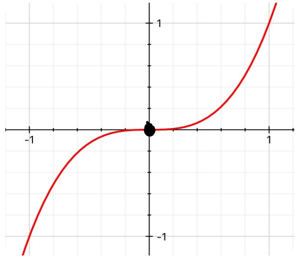
Defn $c \in D_f$ is called a critical point if $f'(c) = 0$ or DNE

Warning: Critical pt may not be rel. max/min.

eg $f(x) = x^3$ $\quad f'(0) = 0$

$\Rightarrow$ 0 is a critical pt.

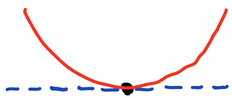
But 0 is neither relative max/min.



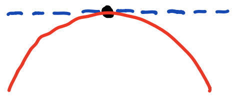
Second Derivative test

Suppose $f'(c) = 0$. If $\begin{cases} f''(c) > 0 \\ f''(c) < 0 \end{cases}$

then f has relative $\begin{cases} \text{minimum} \\ \text{maximum} \end{cases}$ at c



$f'(c) = 0$, $f''(c) > 0$



$f'(c) = 0$, $f''(c) < 0$

Rmk No conclusion if $f'(c) = f''(c) = 0$

eg Let $f(x) = 2x^3 + 3x^2 - 12x - 3$ on $\mathbb{R}$

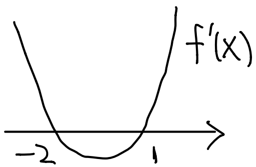
ⓐ Find intervals where $f(x)$ is increasing / decreasing / concave up / concave down

ⓑ Find and classify the critical points of $f(x)$ using
i. 1st Derivative test
ii 2nd Derivative test

ⓒ Find the points of inflection.

Sol

ⓐ $f'(x) = 6x^2 + 6x - 12$
$= 6(x^2 + x - 2)$
$= 6(x+2)(x-1)$



$f'(x) = 0 \iff x = -2$ or 1

	$x < -2$	$-2 < x < 1$	$x > 1$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$
	increasing	decreasing	

$f''(x) = 12x + 6$

$f''(x) = 0 \iff x = -\frac{1}{2}$

	$x < -\frac{1}{2}$	$x > -\frac{1}{2}$
$f''(x)$	$-$	$+$
$f(x)$	$\frown$	$\smile$
	concave down	concave up

f is increasing on $(-\infty, -2]$ and $[1, +\infty)$
decreasing on $[-2, 1]$
concave up on $[-\frac{1}{2}, \infty)$
concave down on $(-\infty, -\frac{1}{2}]$

ⓑ $D_f = \mathbb{R}$, $f'(x)$ exists $\forall x \in \mathbb{R}$

$f'(x) = 0 \iff x = -2$ or 1

$\therefore$ Critical points : -2 and 1

b i By 1st derivative test,

$f'(x) > 0$ on $(-\infty, -2)$
$f'(x) < 0$ on $(-2, 1)$
$\Rightarrow$ f has relative max. at -2

$f'(x) < 0$ on $(-2, 1)$
$f'(x) > 0$ on $(1, \infty)$
$\Rightarrow$ f has relative min. at 1

ii $f''(x) = 12x + 6$

By 2nd derivative test,

$f''(-2) = -18 \Rightarrow f$ has relative max. at -2

$f''(1) = 18 \Rightarrow f$ has relative min at 1

Same conclusion.

c. $f''(-\frac{1}{2}) = 0$

$f''(x) < 0$ for $x < -\frac{1}{2}$

$f''(x) > 0$ for $x > -\frac{1}{2}$

Point of inflection: $(-\frac{1}{2}, f(-\frac{1}{2})) = (-\frac{1}{2}, \frac{7}{2})$

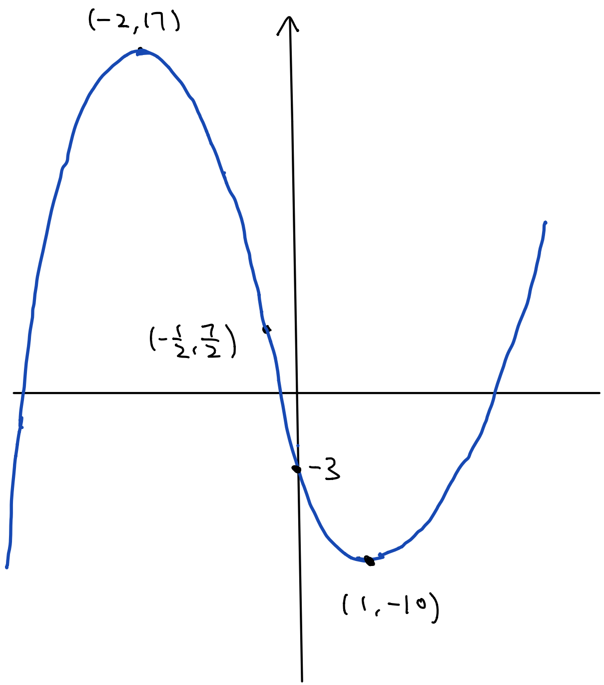
Rmk

One may use this to sketch $y = f(x)$

Relative max: $x = -2$, $f(-2) = 17$

Relative min: $x = 1$ $f(1) = -10$

y-intercept: $f(0) = -3$



Curve Sketching

To graph $y = f(x)$, consider

i. Domain

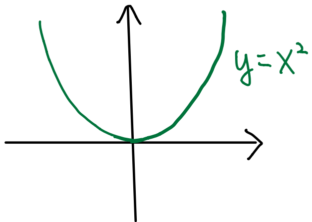
ii. Intercepts

- x-intercept: $\{x \in D_f : f(x) = 0\}$
- y-intercept: $(0, f(0))$, if $0 \in D_f$

iii. Symmetry

- Even function

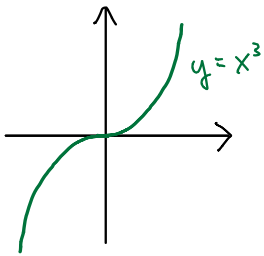
$f(-x) = f(x)\ \forall x \in D_f$



Symmetric about y-axis

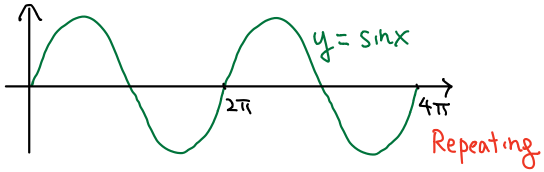
- Odd function

$f(-x) = -f(x)\ \forall x \in D_f$



Symmetric about origin

- Periodic function $f(x+c) = f(x)$, $c \neq 0$, $\forall x$



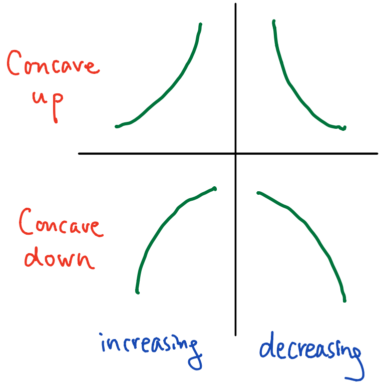
iv. Asymptotes

- Horizontal asymptote: $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x) = c$?
- Vertical asymptote: $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x) = \pm\infty$?

v. Monotonicity (1st Derivative)

- Critical points, Relative/Absolute Extremum
- Increasing: $f'(x) \geq 0$

  Decreasing: $f'(x) \leq 0$



vi. Concavity (2nd Derivative)

- Points of inflection
- Concave up: $f''(x) > 0$
- Concave down: $f''(x) < 0$

eg Graph $f(x) = xe^x$

i Domain
ii Intercepts
iii Symmetry
iv Asymptotes
v. Monotonicity
vi Concavity

i $D_f = \mathbb{R}$

ii $f(0) = 0$

$f(x) = 0 \Leftrightarrow x = 0$

$\therefore (0,0)$ is the only x- or y-intercept

iii $f(-x) \neq \pm f(x)$, f is not periodic

iv. f is continous on $\mathbb{R}$

$\Rightarrow$ no vertical asymptote

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} xe^x = +\infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{e^{-x}} \quad \left(\frac{-\infty}{\infty}\right)$$

$$= \lim_{x\to-\infty} \frac{1}{-e^{-x}} \quad \text{L'Hopital}$$

$$= 0$$

$\therefore y = 0$ is a horizontal asymptote

v. $f'(x) = e^x + xe^x = (x+1)e^x \begin{cases} < 0 & \text{if } x < -1 \\ = 0 & \text{if } x = -1 \\ > 0 & \text{if } x > -1 \end{cases}$

$\Rightarrow$ f is decreasing on $(-\infty, -1]$, increasing on $[-1, +\infty)$

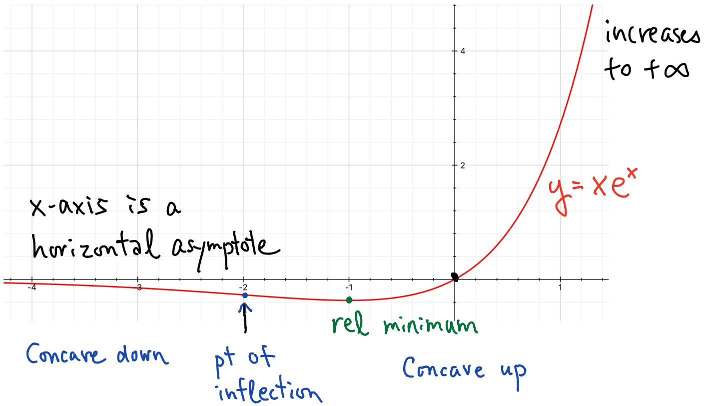
1st Derivative test $\Rightarrow$ f has relative minimum at -1

$f(-1) = -e^{-1} = -\frac{1}{e}$

vi $f''(x) = e^x + e^x + xe^x = (x+2)e^x \begin{cases} < 0 & \text{if } x < -2 \\ = 0 & \text{if } x = -2 \\ > 0 & \text{if } x > -2 \end{cases}$

$\Rightarrow$ f is concave down on $(-\infty, -2)$, concave up on $(-2, +\infty)$

$\therefore (-2, f(-2)) = (-2, -2e^{-2})$ is a point of inflection.



eg Graph $f(x) = \dfrac{1}{x^2+3}$

Sol

- $D_f = \mathbb{R}$
- $f(0) = \frac{1}{3} \Rightarrow$ y-intercept : $(0, \frac{1}{3})$

  $f(x) \neq 0$ for any $x \in \mathbb{R} \Rightarrow$ no x-intercept
- $f(-x) = \dfrac{1}{(-x)^2+3} = \dfrac{1}{x^2+3} = f(x)$

  $\therefore$ $f(x)$ is even function

  $\Rightarrow$ Graph $y = f(x)$ is symmetric about y-axis.
- $D_f = \mathbb{R} \Rightarrow$ no vertical asymptote

  $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = 0$

  $\Rightarrow$ $y = 0$ is a horizontal asymptote

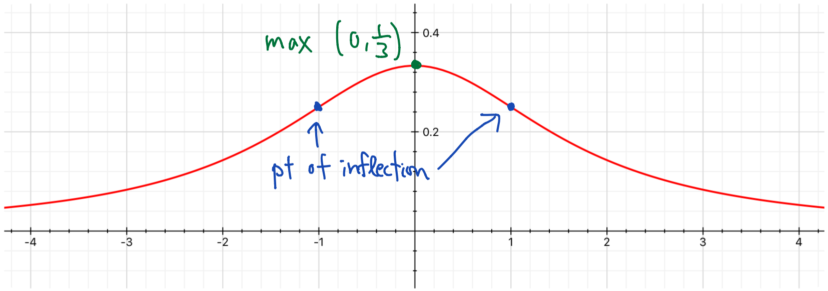
$$f'(x) = \frac{-2x}{(x^2+3)^2}$$

$$f''(x) = \frac{(x^2+3)^2(-2) - (-2x)(2)(x^2+3)(2x)}{(x^2+3)^4} = \frac{6(x^2-1)}{(x^2+3)^3}$$

x	$x<0$	$x=0$	$x>0$
$f'(x)$	$+$	0	$-$
$f(x)$	increasing	max	decreasing

x	$x<-1$	$x=-1$	$-1<x<1$	$x=1$	$x>1$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	Concave up	Pt of inflection	Concave down	Pt of inflection	Concave up

$f(0) = \frac{1}{3}$, $f(-1) = f(1) = \frac{1}{4}$



eg Graph $f(x) = \dfrac{2x^2 - 3x}{x-2}$

Sol

- $D_f = \mathbb{R} \setminus \{2\}$

- $f(0) = 0 \Rightarrow$ y-intercept : $(0,0)$

- $f(x) = 0 \Leftrightarrow x(2x-3) = 0,\ x \neq 2$

$\Leftrightarrow x = 0$ or $\frac{3}{2}$

x-intercepts : $(0,0)$, $(\frac{3}{2}, 0)$

- $f(-x) \neq \pm f(x)$

not even/odd/periodic

- $2 \notin D_f$,

$\lim\limits_{x \to 2^+} f(x) = +\infty \quad \lim\limits_{x \to 2^-} f(x) = -\infty$

$\Rightarrow x = 2$ is a vertical asymptote

Rmk (Not for Exam)

By long division

$$\begin{array}{r} 2x+1 \\ x-2 \,\big)\, \overline{2x^2 - 3x + 0} \\ \underline{2x^2 - 4x} \\ x + 0 \\ \underline{x - 2} \\ 2 \end{array}$$

$2x^2 - 3x = (2x+1)(x-2) + 2$

$\Rightarrow f(x) = 2x + 1 + \dfrac{2}{x-2}$

$f(x) - (2x+1) = \dfrac{2}{x-2}$

$\therefore \lim\limits_{x \to \infty} f(x) - (2x+1) = 0 \quad \lim\limits_{x \to -\infty} f(x) - (2x+1) = 0$

$\Rightarrow y = 2x+1$ is a "slant asymptote"

- $f(x) = 2x + 1 + \dfrac{2}{x-2}$

$f'(x) = 2 - \dfrac{2}{(x-2)^2} \qquad f''(x) = \dfrac{4}{(x-2)^3}$

$f'(x) = 0 \Leftrightarrow 2 - \dfrac{2}{(x-2)^2} = 0 \Leftrightarrow (x-2)^2 = 1$

$\Leftrightarrow x = 1$ or 3

$f''(x) = 0 \Leftrightarrow \dfrac{4}{(x-2)^3} = 0$ (no solution)

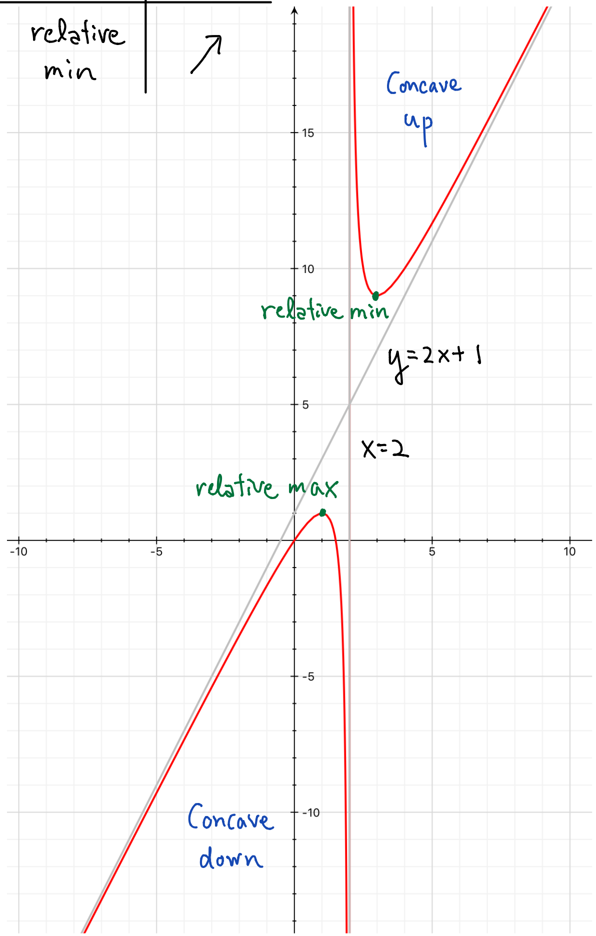
x	$x<1$	$x=1$	$1<x<2$	$x=2$	$2<x<3$	$x=3$	$x>3$
$f'(x)$	+	0	−	/	−	0	+
$f(x)$	↗	relative max	↘	undefined	↘	relative min	↗

$f(1) = 1$

$f(3) = 9$

x	$x<2$	$x=2$	$x>2$
$f''(x)$	−	/	+
$f(x)$	concave down	undefined	concave up

no point of inflection



eg Graph $f(x) = 6x + \sin 2x - 4\cos x$

Sol

- $D_f = \mathbb{R}$
- y-intercept : $(0, f(0)) = (0, -4)$

  x-intercept : Not easy to find ...
- A "little bit" periodic ...

  $f(x) = 6x +$ periodic function
- No asymptote
- $f'(x) = 6 + 2\cos 2x + 4\sin x$

  To solve for $f'(x) = 0$,

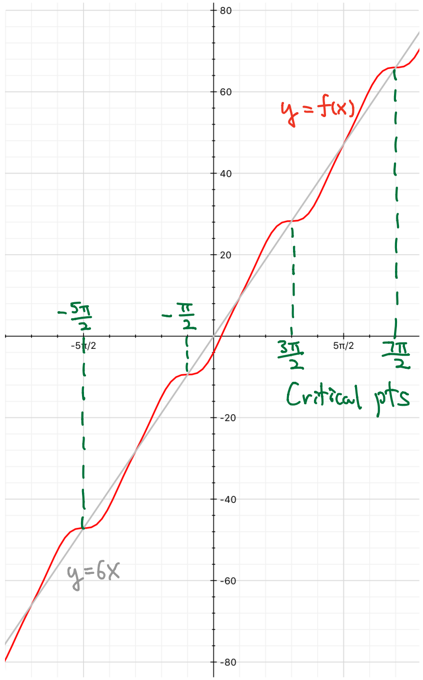
$$\begin{aligned} f'(x) &= 6 + 2(1 - 2\sin^2 x) + 4\sin x \\ &= -4\sin^2 x + 4\sin x + 8 \\ &= -4(\sin^2 x - \sin x - 2) \\ &= -4(\sin x + 1)(\sin x - 2) \end{aligned}$$

Note $\sin x + 1 \geq 0$, $\sin x - 2 \leq 0$ $\forall x \in \mathbb{R}$

$\Rightarrow f'(x) = -4(\sin x + 1)(\sin x - 2) \geq 0$ $\forall x \in \mathbb{R}$

$\therefore f(x)$ is increasing on $(-\infty, \infty)$

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow -4(\sin x + 1)(\sin x - 2) = 0 \\ &\Leftrightarrow \sin x = -1 \quad \text{or} \quad \sin x = 2 \text{ (no solution)} \\ &\Leftrightarrow x = \underbrace{2k\pi - \tfrac{\pi}{2}}_{\text{Stationary}}, \ k \in \mathbb{Z} \end{aligned}$$



Ex Show that inflection points occurs at

$x = 2k\pi + \frac{\pi}{6},\ 2k\pi + \frac{5\pi}{6}$

or $(k + \frac{1}{2})\pi$

Finding absolute max/min.

Recall: By Extreme Value Theorem

f is continuous on $[a,b]$ $\Rightarrow$ f has absolute max and min. on $[a,b]$

Q How to find the absolute extrema?

> **Fact** For a continuous function $f(x)$ on $[a,b]$,
> If f has a relative extrema at c, then c is a critical point or an endpoint, i.e.
> ① $f'(c)=0$ or DNE, or
> ② $c = a$ or b

Abs extremum $\Rightarrow$ Rel. extremum $\Rightarrow$ Critical/End points

Strategy to find absolute max/min

① Find critical points

② Compare values of $f(x)$ at critical points and end points to determine absolute max/min

eg Find the absolute max and min of

$$f(x) = x^{\frac{5}{3}} + 2x^{\frac{2}{3}}$$

on $[-1,1]$.

Sol Note that f is continuous on $[-1,1]$

EVT $\Rightarrow$ f has absolute max and min on $[-1,1]$

To find them ...

① Find the critical points

$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} + \frac{4}{3}x^{-\frac{1}{3}} \quad \text{for } x \neq 0$$

Check if $f'(0)$ exists:

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{\frac{5}{3}} + 2h^{\frac{2}{3}}}{h}$$

$$= \lim_{h\to 0}\left(\underbrace{h^{\frac{2}{3}}}_{\text{approach } 0} + \underbrace{2h^{-\frac{1}{3}}}_{\text{DNE}}\right) \quad \text{DNE}$$

$\therefore$ $f(x)$ is not differentiable at 0

For $f'(x)=0$,

$$\frac{5}{3}x^{\frac{2}{3}}+\frac{4}{3}x^{-\frac{1}{3}}=0$$

$$\frac{x^{-\frac{1}{3}}}{3}(5x+4)=0$$

$$x=-\frac{4}{5}$$

$\therefore$ Critical points are $\underset{f'(x)\text{ DNE}}{0}$, $\underset{f'(x)=0}{-\frac{4}{5}}$

② Compare f(x) at critical points and end points

$f(0)=0$

$f(\frac{4}{5})=(-\frac{4}{5})^{\frac{5}{3}}+2(-\frac{4}{5})^{\frac{2}{3}}\approx 1.034$

$f(-1)=-1+2=1$

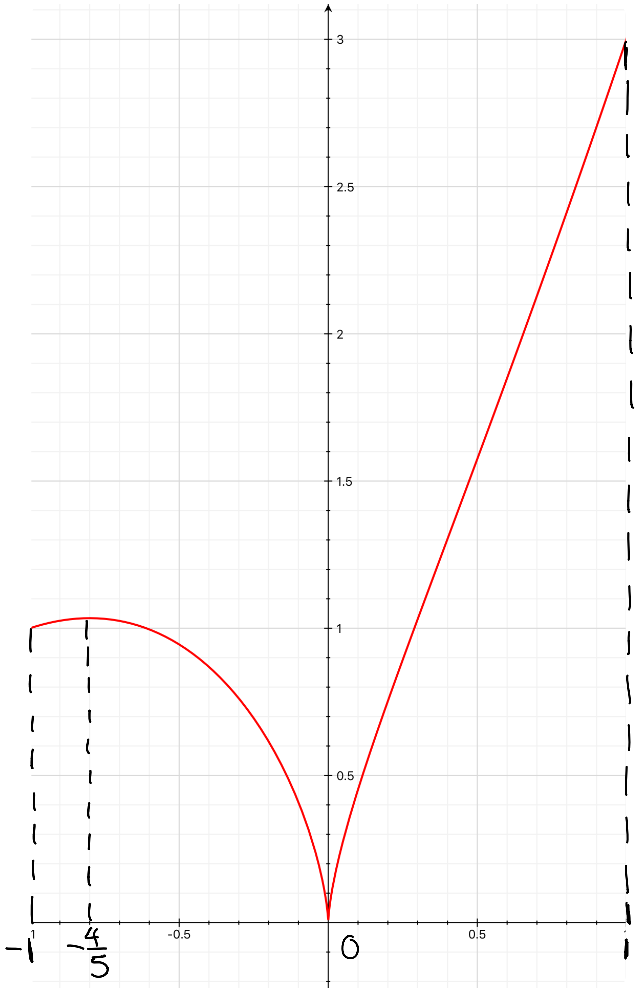
$f(1)=1+2=3$

values at endpoints

$\therefore$ For $-1\le x\le 1$, f has max value 3 at $x=1$

min value 0 at $x=0$

Ex Graph f(x)

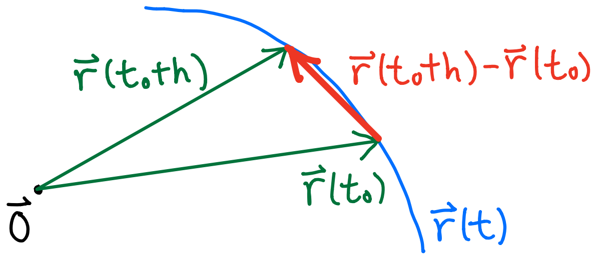


Derivative of vector-valued functions

Let $\vec{r}(t)$ be a vector-valued function.

Its derivative is defined to be

$$\vec{r}\,'(t_0) = \lim_{h \to 0} \frac{\vec{r}(t_0+h) - \vec{r}(t_0)}{h}$$



If $\vec{r}(t)$ = displacement at time t

then $\vec{r}\,'(t)$ = velocity

$\|\vec{r}\,'(t)\|$ = speed

$\vec{r}\,''(t)$ = acceleration

$$\boxed{\begin{aligned} &\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} \\ \Rightarrow\ &\vec{r}\,'(t) = x'(t)\,\hat{i} + y'(t)\,\hat{j} \end{aligned}}$$

eg Let $\vec{r}(t) = (\cos t)\,\hat{i} + (\sin t)\,\hat{j}$ be displacement. Find displacement, velocity, accleration, speed at $t = \frac{\pi}{2}$.

Sol $\vec{r}\,'(t) = (-\sin t)\,\hat{i} + (\cos t)\,\hat{j}$

$\vec{r}\,''(t) = (-\cos t)\,\hat{i} + (-\sin t)\,\hat{j}$

At $t = \frac{\pi}{2}$, displacement $= \vec{r}(\frac{\pi}{2}) = \hat{j}$

velocity $= \vec{r}\,'(\frac{\pi}{2}) = -\hat{i}$

acceleration $= \vec{r}\,''(\frac{\pi}{2}) = -\hat{j}$

speed $= \|\vec{r}\,'(\frac{\pi}{2})\| = \|-\hat{i}\| = 1$

Rmk Note $x = \cos t$, $y = \sin t$

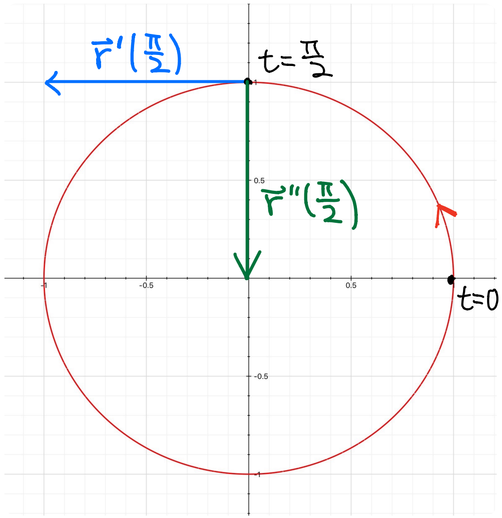
$$\Rightarrow \quad x^2 + y^2 = 1$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2}$$

$$= 1 \quad \text{for any } t$$

$\Rightarrow$ constant speed 1

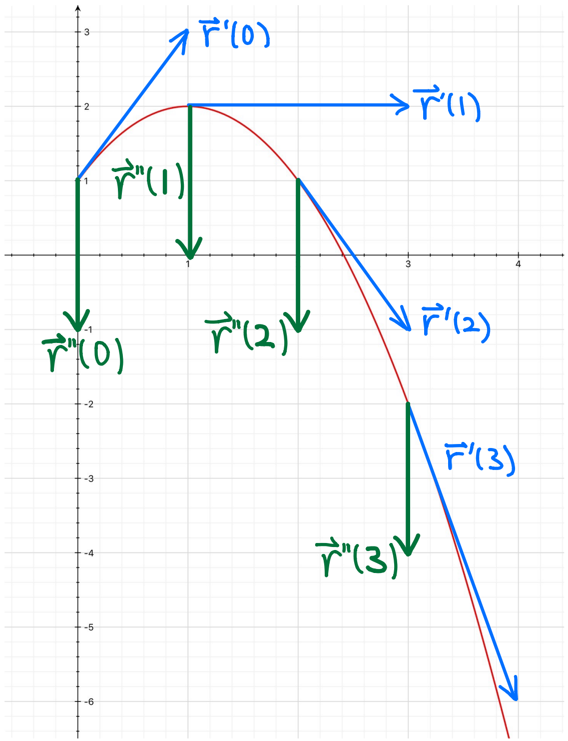
Picture



eg $\vec{r}(t) = t\,\hat{i} + (1 + 2t - t^2)\,\hat{j}$

Then $\vec{r}\,'(t) = \hat{i} + (2 - 2t)\,\hat{j}$

$\vec{r}\,''(t) = -2\,\hat{j}$ (constant acceleration)



Rmk

$x = t$

$y = 1 + 2t - t^2$

$\Rightarrow y = 1 + 2x - x^2$